Nonlocal evolution of weighted scale-free networks

K.-I. Goh, B. Kahng, and D. Kim
School of Physics and Center for Theoretical Physics, Seoul National University NS50, Seoul 151-747, Korea
(Received 3 October 2004; revised manuscript received 31 March 2005; published 26 July 2005)

We introduce the notion of globally updating evolution for a class of weighted networks, in which the weight of a link is characterized by the amount of data packet transport flowing through it. By noting that the packet transport over the network is determined nonlocally, this approach can explain the generic nonlinear scaling between the strength and the degree of a node. We demonstrate by a simple model that the strength-driven evolution scheme recently introduced can be generalized to a nonlinear preferential attachment rule, generating the power-law behaviors in degree and in strength simultaneously.

DOI: 10.1103/PhysRevE.72.017103 PACS number(s): 89.75.Hc, 05.40.−a, 87.23.Kg

Network research has arisen as the interdisciplinary subject for studying complex systems [1–7]. Although the binary (on/off) picture of connectivity has been shown to be quite informative and led to important progress in our understanding of complex systems, such as the ubiquity of power laws in its connectivity pattern, the degree distribution [8], $p_d(k) \sim k^{-\gamma}$, one may need to put a step forward to describe them more realistically. The weighted network, in which links between nodes in the network, or nodes themselves, bear nonuniform weights, is one of the most straightforward generalizations in this direction [9–21]. The weight may represent the intimacy between individuals in social networks, or the bandwidths of routers and optical cables in the Internet. The weighted network can be characterized by the generalized adjacency matrix $\mathbf{W}$ whose element $\{w_{ij}\}$ denotes the weight of the link connecting the node i to j. By definition, $w_{ij}=0$ when there is no link between i and j. We will restrict our interest here to the case of nonnegative weight, $\{w_{ij} \geqslant 0\}$ for all (i,j) [22,23]. The strength s of a node [12], given by

$$s_i = \sum_{j=1}^{N} w_{ij}, \tag{1}$$

generalizes the concept of the degree, the number of links it has in binary networks. In terms of $\{w_{ij}\}$, the degree k of a node may be written as

$$k_i = \sum_{j=1}^{N} \mathrm{sgn}(w_{ij}). \tag{2}$$

The analysis of weighted networks has been hindered mostly by the lack of large-scale data for the real networks. Recently, Barrat *et al.* [12] made the first detailed comparative analysis on the structure of weighted networks of the real world, the scientific coauthorship network (SCN) and the worldwide airport network (WAN). For the SCN, the weight of a link between two scientists is given roughly by the frequency of their collaboration, the effective number of papers they cowrote. For the WAN, the weight is taken as the total number of passengers of the direct flights between two connected cities. Interestingly, the two networks reveal qualitatively different organization of weight and network topology. For the SCN, the strength of a node (scientist) scales linearly with the degree, that is,

$$s(k) \sim k. \tag{3}$$

On the other hand, the WAN exhibits a nonlinear scaling as

$$s(k) \sim k^{\beta}, \tag{4}$$

with $\beta \approx 1.5$.

Later, Barrat, Barthelemy, and Vespignani (BBV) introduced a simple evolution model for such weighted networks [14]. Basically, the BBV model is similar to the Barabási-Albert (BA) model of binary scale-free (SF) network [8] in spirit, containing the growth and the preferential attachment (PA) as basic ingredients. But the difference comes in the aspects that (i) strength at each node and weight at each link are introduced and (ii) the PA rule in the strength instead of the degree is applied. That is, the probability Π_i that an existing node i will receive a connection from a newly introduced node is proportional to its strength, $\Pi_i \sim s_i$. The strength of the target node subsequently increases by a constant amount Δ and so does the weight of the links incident upon the target node in a linear fashion. Then the degree and the strength scales linearly with each other and the distributions of them both follow power laws. On the other hand, the nonlinear scaling Eq. (4) observed in the WAN requires different approaches. BBV proposed the nonlinear coupling between node strength and link weight, but a finite cutoff in node strength was necessary to achieve the SF behavior in both the degree and the strength distribution [24].

In this Brief Report, we propose a packet transport-driven evolution model of weighted SF network, which can illustrate the nonlinear relationship Eq. (4). We first point out that the weight used for the WAN in Ref. [12] is actually the amount of traffic through the link. From this perspective, the evolution of such weighted network should be viewed as packet transport-driven. The distinguishing point here is that the traffic flowing through the network is determined in a nonlocal manner, in high contrast to the local weight evolution rule as formulated in the BBV model [14]. A measure of such traffic over the (binary) network is the quantity called the load [25], or the betweenness centrality [26]. The load of a node, the vertex load, is defined by the effective number of

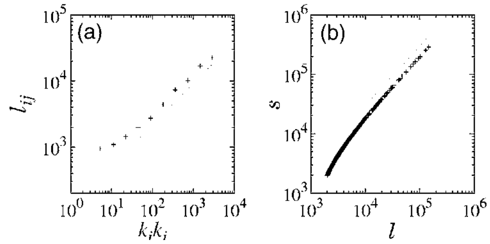

FIG. 1. (a) Link load ℓ_{ij} vs the product of the two degrees k_ik_j of the nodes at each end of the link in the BA model with the system size $N=10^3$. Data points are logarithmically binned to reduce fluctuation. The straight line with slope 0.63 is drawn for reference; (b) relation between the node strength (the sum of the link loads attached to the node) and the load of the node. As indicated by the guideline with slope 1, they scale linearly for large ℓ.

data units passing through that node when every pair of nodes in the network sends and receives unit data in unit time step. The data are assumed to be delivered only along the shortest path(s) between the pair. When the data encounter a branching point during the transport, they are supposed to be divided evenly by the number of branches. Thus the load quantifies the level of traffic, albeit in an idealized way. One can also define the link load in a similar fashion, as the effective number of data units passing through a given link. For SF networks, the load of each node is heterogeneous, and its distribution follows a power law, $p_L(\ell) \sim \ell^{-\delta}$ [25,27]. If the ranks of a node in degree and in load are preserved, then the scaling relation,

$$\ell_i \sim k_i^{\eta}, \tag{5}$$

holds for each node i, where $\eta=(\gamma-1)/(\delta-1)$. This relation is valid in the BA-type model and the Internet [28].

The weight of a link w_{ij} and the product of the degrees of the nodes at its ends k_ik_j are found to be related as a power law, $w_{ij} \sim (k_ik_j)^{\theta}$, and $\theta \approx 1/2$ for the WAN [12]. Interestingly, the same half-power scaling was observed in the metabolic network of *Escherichia coli*, between the metabolic flux of a reaction and the degrees of the participating metabolites [18]. It is worthwhile to note that such a correlation has also been observed in the relation of the link load ℓ_{ij} versus k_ik_j for binary networks [18,29]. We show in Fig. 1(a) such a relation for the BA model [8], finding a slightly different scaling exponent $\theta \approx 0.63$. So we may regard the link load in the binary network as an approximation of the weight in the weighted networks characterized by the traffic level, such as the WAN. Although actual traffic level may well depend also on more complicated factors such as the geographic distances between the airports and the queuing and transit delays at the airports, we exploit this idea as a starting point for further discussion below. In this setting, the strength of a node is given approximately by the load of the node itself, i.e.,

$$s_i = \sum_{j \neq i} \ell_{ij} \sim \ell_i, \tag{6}$$

since the vertex load of a node is roughly one half of the sum of the link loads of the links connected to that node for large ℓ as shown in Fig. 1(b).

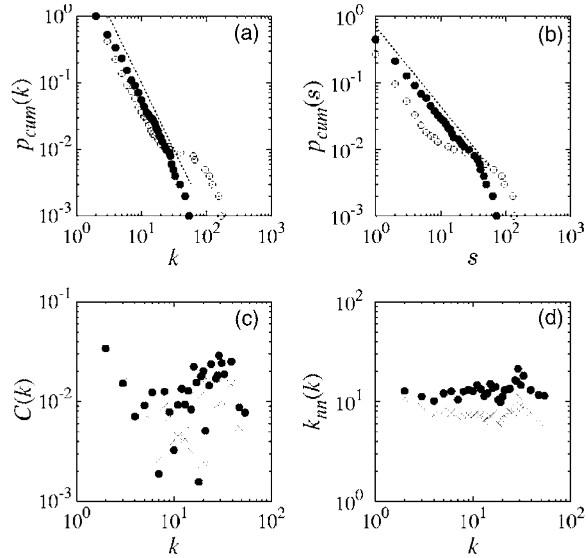

FIG. 2. Cumulative distributions for the degree (a) and the strength (b) of the packet transport-driven network growth with Eq. (7). The open circles denote the data for $\alpha=1$ and the filled circles for $\alpha=0.6$. The guidelines have slopes -2.0 (a) and -1.2 (b). Note, the humps for large degrees and large strengths in the case of $\alpha=1$, indicating the breakdown of the SF nature. The local clustering function $C(k)$ (c) and the average neighbor degree function $k_{nn}(k)$ (d) for the network with $\alpha=0.6$. The filled circles denote the weighted version of the corresponding quantities introduced in Ref. [12] and the cross symbols the original binary version of them.

When the relation (5) holds, the strength and the degree of a given node scale nonlinearly as $s_i \sim k_i^{\eta}$. Then the application of BBV scheme leads to $\Pi_i \sim s_i \sim k_i^{\eta}$ with $\eta > 1$ in most cases. Such a super-PA in degree breaks the power-law degree distribution [30] as well as Eq. (5) itself. To obtain a SF network, one must use the sub-PA in strength,

$$\Pi_i \sim s_i^{\alpha} \sim \ell_i^{\alpha} \tag{7}$$

with $\alpha = 1/\eta$. At a first glance, one may think that this rule of sub-PA in strength would not generate a SF strength distribution, however, we can achieve a SF strength distribution with the strength being the load.

To demonstrate the above idea, we simulate a growing network as follows. (i) Initially, $m_0=3$ nodes are introduced and they are fully connected, and we calculate the load at each node. (ii) At each time step, a new node is added, and attaches $m=2$ links to existing nodes selected following the PA rule $\Pi_i \sim \ell_i^{\alpha}$. (iii) The load of each node is recalculated. This process is repeated N times. Figures 2(a) and 2(b) show the simulation results, the degree and the strength (or load) distributions of the packet transport-driven growth network for the cases of $\alpha_1=1$ and $\alpha_2=1/\eta \approx 0.6$. Indeed we can see that for the $\alpha=1$ case, the power-law behaviors of the degree and the strength distributions break down, while for $\alpha=0.6$, the model reproduces both the power laws with the degree exponent $\gamma \approx 3$ and the load exponent $\delta \approx 2.2$, equivalent to those of the BA model. These measured values of γ and δ are consistent with $\alpha=0.6$ through the relation $\alpha=(\delta-1)/(\gamma-1)$. Other structural properties such as the clustering and the

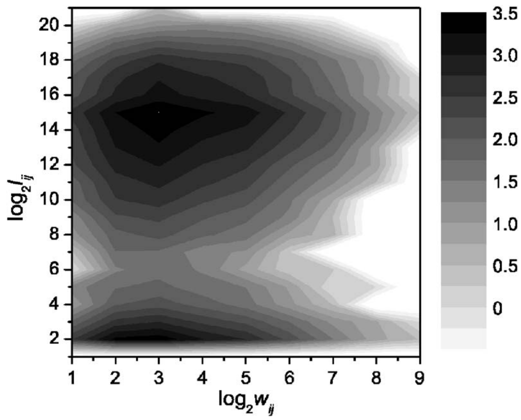

FIG. 3. Histogram of the links having the link load ℓ_{ij} and the link weight w_{ij} in the coauthorship network of cond-mat subset of arXiv.org [31]. The gray level denotes the number of links in each bin, in logarithm with base 10.

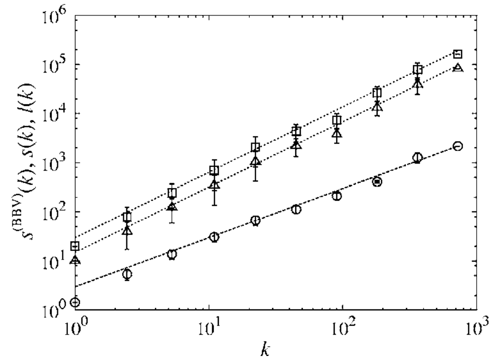

FIG. 4. Strength $s(k)$ based on the link load (□), the load $\ell(k)$ based on the vertex load (△), and the BBV strength $s^{(\mathrm{BBV})}(k)$ (○) vs the degree k for the BBV model network with system size $N=10^4$, the number of links emanating from a new node $m=1$, and the weight increment $w_0=\Delta=1.0$. The slopes of the guide lines are 1.33 (dotted) and 1.0 (dashed), respectively.

degree mixing are also similar to those of the BA model [Figs. 2(c) and 2(d)]. It is of no surprise because this model with $\alpha=1/\eta$ is nothing but the BA model rephrased in terms of the strength-driven evolution.

Note that in our model, the link weight update is carried out through the change of the load, which occurs by global reorganization of shortest pathways within the network. As pointed out earlier, the load accounts for the traffic level only in an approximated manner. The general framework described in the paper, however, does not depend on what to use as the traffic measure as long as the scaling relation Eq. (4) holds for that quantity.

Surely, it is not always the case that the weight of a link is determined by the level of traffic. In the SCN, for example, the weight measures the direct affinity between the scientists, which is primarily determined by their own attributes which is local in character. Indeed, for SCN we can see no appreciable correlation between the weight and the link load of a link, as shown in Fig. 3. In such a case, the BBV scheme could be more suitable than ours. Thus in the weighted network modeling, one has first to discriminate properly what the nature of the weight is in the system one wants to describe.

Finally, we like to emphasize again the difference between the strength used here and that of the BBV model. We also compare them with the vertex load directly measured. For the purpose, we measure the strength s_i of node i based on Eq. (1) where the weight w_{ij} is given as the link load ℓ_{ij}, that is, $s_i=\Sigma_j \ell_{ij}$, and compare it with the vertex load ℓ_i measured directly from the BBV network and the quantity $s_i^{(\mathrm{BBV})}$ defined in the BBV model [14]. We note that s_i and ℓ_i ($s_i^{(\mathrm{BBV})}$) are updated globally (locally) when a new node is added in the system. As shown in Fig. 4, we can find the scaling behaviors of $s \sim \ell \sim k^{1.33}$, which is consistent with the one obtained from the formula, $\ell \sim k^{(\gamma-1)/(\delta-1)}$, and $s^{(\mathrm{BBV})} \sim k$. Those results support our claim.

To conclude, we have introduced the notion of globally updating evolution in a class of weighted networks, in which the weight is characterized by the level of traffic flowing through the links. The link load is used as a measure of the traffic in our model. This model explains the generic nonlinear scaling between the strength (traffic) and the degree, observed in, e.g., the WAN. We have also shown that the generalization of strength-driven evolution into the nonlinear PA in strength is necessary to produce a network which is SF both in degree and in strength.

Note added: Recently we have learned of a recent work by Bianconi [32] which introduced a model generating the nonlinear scaling relation between strength and degree, Eq. (4). The model succeeds to obtain the nonlinear relationship by decoupling the evolution of the network topology (degree) and the weights. Although that model seems to be different from our conceptual model at first, the two models share important aspects. First its binary network structure grows following the BA model and second, the weights are updated nonlocally rather than locally as was assumed in the BBV model. Moreover, Guimerà and Amaral [33] introduced a model to illustrate the traffic in the WAN. The model also contains the rule of adding internal links between existing nodes, which are chosen by the PA rule combined with the ingredient of reducing traveling length. Such a rule is applied to all nodes in the network, not limited to neighbors of a newly added node, which needless to say accounts for global updating of traffic pathways. Such global reorganization of weights produces the nonlinear relationship between strength and degree, which is the main conclusion of the work.

We would like to thank Mark Newman for making the arXiv.org coauthorship network data available. This work is supported by the KOSEF Grant No. R14-2002-059-01000-0 in the ABRL Program.

[1] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[2] S. N. Dorogovtsev and J. F. F. Mendes, *Evolution of Networks* (Oxford University Press, Oxford, 2003).
[3] M. E. J. Newman, SIAM Rev. **45**, 167 (2003).
[4] R. Pastor-Satorras and A. Vespignani, *Evolution and Structure of Internet* (Cambridge University Press, Cambridge, 2004).
[5] A.-L. Barabási and Z. N. Oltvai, Nat. Genet. **5**, 101 (2004).
[6] E. V. Koonin, Y. I. Wolf, and G. P. Karev, *Power Laws, Scale-free Networks and Genome Biology* (Landes Bioscience, Georgetown, 2004).
[7] D. J. Watts, Annu. Rev. Sociol. **30**, 243 (2004).
[8] A.-L. Barabási and R. Albert, Science **286**, 509 (1999).
[9] S. H. Yook, H. Jeong, A.-L. Barabási, and Y. Tu, Phys. Rev. Lett. **86**, 5835 (2001).
[10] J. D. Noh and H. Rieger, Phys. Rev. E **66**, 066127 (2002).
[11] L. A. Braunstein, S. V. Buldyrev, R. Cohen, S. Havlin, and H. E. Stanley, Phys. Rev. Lett. **91**, 168701 (2003).
[12] A. Barrat, M. Barthelemy, R. Pastor-Satorras, and A. Vespignani, Proc. Natl. Acad. Sci. U.S.A. **101**, 3747 (2004).
[13] E. Almaas, B. Kovacs, T. Vicsek, Z. N. Oltvai, and A.-L. Barabási, Nature (London) **427**, 839 (2004).
[14] A. Barrat, M. Barthelemy, and A. Vespignani, Phys. Rev. Lett. **92**, 228701 (2004).
[15] W. Li and X. Cai, Phys. Rev. E **69**, 046106 (2004).
[16] K. Park, Y.-C. Lai, and N. Ye, Phys. Rev. E **70**, 026109 (2004).
[17] C. Li and G. Chen, cond-mat/0311333 (2003).
[18] P. J. Macdonald, E. Almaas, and A.-L. Barabási, cond-mat/0405688 (2004).
[19] E. Almaas, P. L. Krapivsky, and S. Redner, Phys. Rev. E **71**, 036124 (2005).
[20] T. Antal and P. L. Krapivsky, Phys. Rev. E **71**, 026103 (2005).
[21] S. N. Dorogovtsev and J. F. F. Mendes, cond-mat/0408343 (2004).
[22] M. E. J. Newman, Phys. Rev. E **70**, 056131 (2004).
[23] J.-P. Onnela, J. Saramäki, J. Kertesz, and K. Kaski, Phys. Rev. E **71**, 065103 (2005).
[24] A. Barrat, M. Barthelemy, and A. Vespignani, Phys. Rev. E **70**, 066149 (2004).
[25] K.-I. Goh, B. Kahng, and D. Kim, Phys. Rev. Lett. **87**, 278701 (2001).
[26] L. C. Freeman, Sociometry **40**, 35 (1977).
[27] K.-I. Goh, E. Oh, H. Jeong, B. Kahng, and D. Kim, Proc. Natl. Acad. Sci. U.S.A. **99**, 12583 (2002).
[28] K.-I. Goh, E. Oh, B. Kahng, and D. Kim, Phys. Rev. E **67**, 017101 (2003).
[29] P. Holme, B. J. Kim, C. N. Yoon, and S. K. Han, Phys. Rev. E **65**, 056109 (2004).
[30] P. L. Krapivsky and S. Redner, Phys. Rev. E **63**, 066123 (2001).
[31] M. E. J. Newman, Proc. Natl. Acad. Sci. U.S.A. **98**, 404 (2001).
[32] G. Bianconi, cond-mat/0412399 (2004).
[33] R. Guimerà and L. A. N. Amaral, Eur. Phys. J. B **38**, 381 (2004).